REFINEMENTS OF A REVERSED AM–GM OPERATOR INEQUALITY

MOJTABA BAKHERAD

Abstract. We prove some refinements of a reverse AM–GM operator inequality due to M. Lin [Studia Math. 2013;215:187–194]. In particular, we show the operator inequality

$$\Phi^p\left(A\nabla_\nu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \leq \alpha^p \Phi^p\left(A\sharp_\nu B\right),$$

where A, B are positive operators on a Hilbert space such that $0 < m \leq A, B \leq M$ for some positive numbers m, M, Φ is a positive unital linear map, $\nu \in [0,1]$, $r = \min\{\nu, 1-\nu\}$, $p > 0$ and $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$, with the identity I. In the case when $\dim\mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n\times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$, and we then write $A \geq 0$. We write $A > 0$ if A is a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ we say that $A \leq B$ if $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and $I_{\mathscr{H}}$. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$.

Let $A, B \in \mathbb{B}(\mathscr{H})$ be two positive invertible operators and $\nu \in [0,1]$. The operator weighted arithmetic, geometric and harmonic means are defined by $A\nabla_\nu B = (1-\nu)A + \nu B$, $A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^\nu A^{\frac{1}{2}}$ and $A!_\nu B = ((1-\nu)A^{-1} + \nu B^{-1})^{-1}$, respectively. In particular, for $\nu = \frac{1}{2}$ we get the usual operator arithmetic mean ∇,

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the geometric mean $\sharp$ and the harmonic mean $!$. The AM–GM inequality reads

$$\frac{A+B}{2} \geq A\sharp B,$$

for all positive operators A, B. It is shown in [10] the following reverse of AM–GM inequality involving positive linear maps

$$\Phi\left(\frac{A+B}{2}\right) \leq \frac{(M+m)^2}{4Mm}\Phi(A\sharp B), \tag{1.1}$$

where $0 < m \leq A, B \leq M$ and Φ is a positive unital linear map.

For two positive operators $A, B \in \mathbb{B}(\mathscr{H})$, the Löwner–Heinz inequality states that, if $A \leq B$, then

$$A^p \leq B^p, \qquad (0 \leq p \leq 1). \tag{1.2}$$

In general (1.2) is not true for $p > 1$. Lin [10, Theorem 2.1] showed however a squaring of (1.1), namely that the inequality

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi^2(A\sharp B) \tag{1.3}$$

as well as

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 (\Phi(A)\sharp\Phi(B))^2 \tag{1.4}$$

hold. Using inequality (1.2) we therefore get

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p \Phi^p(A\sharp B) \qquad (0 < p \leq 2) \tag{1.5}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p \qquad (0 < p \leq 2), \tag{1.6}$$

where $0 < m \leq A, B \leq M$ and Φ is a positive unital linear map.

In [13] the authors extended (1.3) and (1.4) to $p > 2$. They proved that the inequalities

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p \Phi^p(A\sharp B) \qquad (p > 2) \tag{1.7}$$

and

$$\Phi^p\left(\frac{A+B}{2}\right) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right)^p (\Phi(A)\sharp\Phi(B))^p \qquad (p > 2), \tag{1.8}$$

where $0 < m \leq A, B \leq M$. In [4] and [12] the authors showed that

$$\Phi^p\left(A\sigma B\right) \leq \alpha^p \Phi^p\left(A\tau B\right), \tag{1.9}$$

and

$$\Phi^p (A\sigma B) \le \alpha^p \left(\Phi(A)\tau\Phi(B)\right)^p, \tag{1.10}$$

where $0 < m \le A, B \le M$, Φ be a positive unital linear map, σ, τ be two arbitrary means between harmonic and arithmetic means, $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$ and $p > 0$. Choi's inequality (see e.g. [1, p. 41]) reads

$$\Phi(A)^{-1} \le \Phi(A^{-1}), \tag{1.11}$$

for any positive unital linear map Φ and operator $A > 0$. Choi's inequality cannot be squared [10], but a reverse of Choi's inequality (known as the operator Kantorovich inequality) can be squared, see e.g. [11].

In this paper, we present some refinements of inequalities (1.5) and (1.6) under some mild conditions for $0 < p \le 1$ and some refinements of inequalities (1.7) and (1.8) for the operator norm and $p > 2$. We also show a refinement of the operator Pólya–Szegö inequality.

2. Main results

We need the following lemmas to prove our results.

Lemma 2.1. [3] *Let $A, B > 0$. Then*

$$\|AB\| \le \frac{1}{4}\|A+B\|^2.$$

Lemma 2.2. [8] *Let $A, B \ge 0$ and $p > 1$. Then*

$$\|A^p + B^p\| \le \|(A+B)^p\|.$$

Lemma 2.3. *Let $A, B > 0$ and $\alpha > 0$. Then $A \le \alpha B$ if and only if $\|A^{\frac{1}{2}}B^{\frac{-1}{2}}\| \le \alpha^{\frac{1}{2}}$.*

Proof. Obviously, $A \le \alpha B$ if and only if $B^{\frac{-1}{2}}AB^{\frac{-1}{2}} \le \alpha$. By definition, this holds if and only if $\|A^{\frac{1}{2}}B^{\frac{-1}{2}}\|^2 \le \alpha$ and the proof is complete. □

Lemma 2.4. [4] *Let $0 < m \le A, B \le M$, Φ be a positive unital linear map and σ, τ be two arbitrary means between harmonic and arithmetic means. Then*

$$\Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \le M + m.$$

In the next proposition we extend the inequalities (1.9) and (1.10) to $p > 2$ and the inequalities (1.7) and (1.8) to arbitrary means between harmonic and arithmetic means.

Proposition 2.5. *Let $0 < m \leq A, B \leq M$, Φ be a positive unital linear map, σ, τ be two arbitrary means between harmonic and arithmetic means and $p > 0$. Then*

$$\Phi^p\left(A\sigma B\right)\Phi^{-p}\left(A\tau B\right)+\Phi^{-p}\left(A\tau B\right)\Phi^p\left(A\sigma B\right)\leq 2\alpha^p$$

where $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{1}{p}}Mm}\right\}$.

Proof. By [5, Lemma 3.5.12] we have that $\|X\| \leq t$ if and only if $\begin{pmatrix} tI & X \\ X^* & tI \end{pmatrix} \geq 0$, for any $X \in \mathbb{B}(\mathscr{H})$. If $0 < p \leq 1$, then $\alpha = \frac{(M+m)^2}{4Mm}$. Applying inequality (1.9) and Lemma 2.3 we get

$$\|\Phi^p\left(A\sigma B\right)\Phi^{-p}\left(A\tau B\right)\| \leq \alpha^p.$$

Hence

$$\begin{pmatrix} \alpha^p I & \Phi^p\left(A\sigma B\right)\Phi^{-p}\left(A\tau B\right) \\ \Phi^{-p}\left(A\tau B\right)\Phi^p\left(A\sigma B\right) & \alpha^p I \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} \alpha^p I & \Phi^{-p}\left(A\tau B\right)\Phi^p\left(A\sigma B\right) \\ \Phi^p\left(A\sigma B\right)\Phi^{-p}\left(A\tau B\right) & \alpha^p I \end{pmatrix} \geq 0.$$

Hence

$$\begin{pmatrix} 2\alpha^p I & \Phi^{-p}\left(A\tau B\right)\Phi^p\left(A\sigma B\right)+\Phi^p\left(A\sigma B\right)\Phi^{-p}\left(A\tau B\right) \\ \Phi^p\left(A\sigma B\right)\Phi^{-p}\left(A\tau B\right)+\Phi^{-p}\left(A\tau B\right)\Phi^p\left(A\sigma B\right) & 2\alpha^p I \end{pmatrix}$$

is positive and the desired inequality for $0 < p \leq 1$. Using inequality (1.9) with the same argument, we get the desired inequality for $p > 1$. □

Now, we are ready to present our main result. We need the following lemma, proved in [7]; (see also [2]).

Lemma 2.6. [7] *Let $a, b > 0$ and $\nu \in [0, 1]$. Then*

$$a^{1-\nu}b^\nu + r(\sqrt{a}-\sqrt{b})^2 \leq (1-\nu)a+\nu b, \tag{2.1}$$

where $r = \min\{\nu, 1-\nu\}$.

Theorem 2.7. *Let $0 < m \leq A, B \leq M$, Φ be a positive unital linear map, $\nu \in [0, 1]$ and $p > 0$. Then*

$$\Phi^p\left(A\nabla_\nu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \leq \alpha^p\Phi^p\left(A\sharp_\nu B\right) \tag{2.2}$$

and

$$\Phi^p\left(A\nabla_\nu B + 2rMm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \leq \alpha^p\left(\Phi\left(A\right)\sharp_\nu\Phi\left(B\right)\right)^p, \tag{2.3}$$

where $r = \min\{\nu, 1-\nu\}$ *and* $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$.

Proof. We prove first the inequalities (2.2) and (2.3) for $0 < p \leq 2$. Since $0 < m \leq A, B \leq M$ we get that

$$A + MmA^{-1} \leq M + m \quad \text{and} \quad B + MmB^{-1} \leq M + m.$$

Therefore, for a positive unital linear map Φ we have

$$\Phi(A) + Mm\Phi(A^{-1}) \leq M + m$$

and

$$\Phi(B) + Mm\Phi(B^{-1}) \leq M + m.$$

Obviously we have also the inequalities

$$\Phi((1-\nu)A) + Mm\Phi((1-\nu)A^{-1}) \leq (1-\nu)M + (1-\nu)m$$

and

$$\Phi(\nu B) + Mm\Phi(\nu B^{-1}) \leq \nu M + \nu m.$$

for any $\nu \in [0, 1]$. Summing up, we therefore get

$$\Phi(A\nabla_\nu B) + Mm\Phi((1-\nu)A^{-1} + \nu B^{-1}) \leq M + m. \tag{2.4}$$

Moreover, by using the inequality (2.1) and functional calculus for the positive operator $A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ we have

$$\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{\nu} + r\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} + 1 - 2\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) \leq (1-\nu) + \nu A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}.$$

Multiplying both sides of the above inequality both to the left and to the right by $A^{\frac{-1}{2}}$ we get that

$$A^{-1}\sharp_\nu B^{-1} + 2r\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right) \leq (1-\nu)A^{-1} + \nu B^{-1}. \tag{2.5}$$

Applying (1.11), (2.4), (2.5) and taking into account the properties of Φ we have

$$\begin{aligned}
&\left\|\Phi\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)Mm\Phi^{-1}(A\sharp_\nu B)\right\| \\
&\le \frac{1}{4}\left\|\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\Phi^{-1}(A\sharp_\nu B)\right\|^2 \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.1)} \\
&\le \frac{1}{4}\left\|\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\Phi\big(A^{-1}\sharp_\nu B^{-1}\big)\right\|^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by inequality (1.11))} \\
&= \frac{1}{4}\left\|\Phi(A\nabla_\nu B)+Mm\Phi(A^{-1}\sharp_\nu B^{-1}+2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))\right\|^2 \\
&\le \frac{1}{4}\left\|\Phi(A\nabla_\nu B)+Mm\Phi((1-\nu)A^{-1}+\nu B^{-1})\right\|^2 \quad \text{(by inequality (2.5))} \\
&\le \frac{1}{4}(M+m)^2 \qquad\qquad \text{(by inequality (2.4))}.
\end{aligned}$$

Therefore

$$\left\|\Phi\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)\Phi^{-1}(A\sharp_\nu B)\right\| \le \frac{(M+m)^2}{4Mm}. \tag{2.6}$$

Hence

$$\Phi^2\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^2\Phi^2\left(A\sharp_\nu B\right).$$

Since $0<p/2\le 1$, by inequality (1.2) we have

$$\Phi^p\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right) \le \left(\frac{(M+m)^2}{4Mm}\right)^p\Phi^p\left(A\sharp_\nu B\right).$$

Thus we get the inequality (2.2) for $0<p\le 2$. We prove now (2.3) for $0<p\le 2$. Applying Lemma 2.1 and then inequality (2.2) we have

$$\begin{aligned}
&\left\|\Phi\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)Mm(\Phi(A)\sharp_\nu\Phi(B))^{-1}\right\| \\
&\le \frac{1}{4}\left\|\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm(\Phi(A)\sharp_\nu\Phi(B))^{-1}\right\|^2 \\
&\qquad\qquad \text{(by Lemma 2.1)} \\
&\le \frac{1}{4}\left\|\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\Phi^{-1}(A\sharp_\nu B)\right\|^2 \\
&\le \frac{1}{4}(M+m)^2 \qquad \text{(by inequality (2.6))}.
\end{aligned}$$

Hence the inequality (2.3) for $0<p\le 2$.

Now, we prove the inequalities (2.2) and (2.3) for $p>2$. Then, by Lemma 2.1 and

2.2 we get

$$\begin{aligned}
&M^{\frac{p}{2}}m^{\frac{p}{2}}\left\|\Phi^{\frac{p}{2}}\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)\Phi^{\frac{-p}{2}}(A\sharp_\nu B)\right\| \\
&\quad=\left\|\Phi^{\frac{p}{2}}\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{\frac{-p}{2}}(A\sharp_\nu B)\right\| \\
&\quad\le\frac{1}{4}\left\|\Phi^{\frac{p}{2}}(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{\frac{-p}{2}}(A\sharp_\nu B)\right\|^2 \\
&\quad\le\frac{1}{4}\left\|\left(\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\Phi^{-1}(A\sharp_\nu B)\right)^{\frac{p}{2}}\right\|^2 \\
&\quad=\frac{1}{4}\left\|\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\Phi^{-1}(A\sharp_\nu B)\right\|^p \\
&\quad\le\frac{1}{4}(M+m)^p.
\end{aligned}$$

Hence we get the inequality (2.2) for $p>2$. Further, we have

$$\begin{aligned}
&M^{\frac{p}{2}}m^{\frac{p}{2}}\left\|\Phi^{\frac{p}{2}}\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)\left(\Phi\left(A\right)\sharp_\nu\Phi\left(B\right)\right)^{\frac{-p}{2}}\right\| \\
&\quad=\left\|\Phi^{\frac{p}{2}}\left(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)M^{\frac{p}{2}}m^{\frac{p}{2}}\left(\Phi\left(A\right)\sharp_\nu\Phi\left(B\right)\right)^{\frac{-p}{2}}\right\| \\
&\quad\le\frac{1}{4}\left\|\Phi^{\frac{p}{2}}(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+M^{\frac{p}{2}}m^{\frac{p}{2}}\left(\Phi\left(A\right)\sharp_\nu\Phi\left(B\right)\right)^{\frac{-p}{2}}\right\|^2 \\
&\quad\le\frac{1}{4}\left\|\left(\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\left(\Phi\left(A\right)\sharp_\nu\Phi\left(B\right)\right)^{-1}\right)^{\frac{p}{2}}\right\|^2 \\
&\quad=\frac{1}{4}\left\|\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\left(\Phi\left(A\right)\sharp_\nu\Phi\left(B\right)\right)^{-1}\right\|^p \\
&\quad\le\frac{1}{4}\left\|\Phi(A\nabla_\nu B+2rMm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}))+Mm\Phi^{-1}(A\sharp_\nu B)\right\|^p \\
&\quad\le\frac{1}{4}(M+m)^p.
\end{aligned}$$

Thus we get the inequality (2.3) for $p>2$ and this completes the proof of the theorem. □

Remark 2.8. Let $0<m\le A,B\le M$, Φ be a positive unital linear map. If $0<p\le 1$, then, obviously,

$$\Phi^p\left(A\nabla_\nu B\right)\le\left(\Phi\left(A\nabla_\nu B\right)+2rMm\Phi\left(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1}\right)\right)^p. \tag{2.7}$$

Hence the inequality (2.7) shows that Theorem 2.7 is a refinement of inequalities (1.5) and (1.6) for $0<p\le 1$.

We also have

$$\Phi^p\left(A\nabla_\nu B\right) \le \Phi^p\left(A\nabla_\nu B\right) + (2rMm)^p\Phi^p\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right),$$

where $p \ge 1$, $\nu \in [0,1]$ and $r = \min\{\nu, 1-\nu\}$.

Hence

$$\begin{aligned}\|\Phi^p\left(A\nabla_\nu B\right)\| &\le \left\|\Phi^p\left(A\nabla_\nu B\right) + (2rMm)^p\Phi^p\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right\| \\ &\le \left\|\Phi^p\Big(A\nabla_\nu B + 2rMmA^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\Big)\right\| \qquad \text{(by Lemma 2.2)}.\end{aligned}$$

Therefore, Theorem 2.7 is a refinement of the inequalities, (1.7) and (1.8) for the operator norm and $p \ge 2$.

The following examples show that inequality (2.2) is a refinement of (1.5) and (1.7).

Example 2.9. If $A = \begin{pmatrix} 1.75 & 0.433 \\ 0.433 & 1.25 \end{pmatrix}$, $B = \begin{pmatrix} 2.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix}$, $\Phi(X) = \frac{1}{2}\mathrm{tr(X)}$ $(\mathrm{X} \in \mathbb{M}_2)$, $m = 1$, $M = 3$, $\nu = \frac{1}{2}$ and $p = 3$, then $A\nabla_\nu B = \begin{pmatrix} 2.1250 & 0.4665 \\ 0.4665 & 1.8750 \end{pmatrix}$ and $A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right) = \begin{pmatrix} 2.1601 & 0.4260 \\ 0.4260 & 2.0016 \end{pmatrix}$. Hence

$$\Phi^3\left(A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) - \Phi^3(A\nabla_\nu B) = 9.0095 - 8 = 1.0095 > 0.$$

Example 2.10. Let $\Phi(X) = T^*XT$ $(X \in \mathbb{M}_2)$, where $T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$. If $A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 4.75 & 0.433 \\ 0.433 & 4.25 \end{pmatrix}$, $m = 3$, $M = 7$, $\nu = \frac{1}{2}$ and $p = \frac{5}{3}$, then $A\nabla_\nu B = \begin{pmatrix} 4.8750 & -0.7835 \\ -0.7835 & 4.6250 \end{pmatrix}$ and $A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right) = \begin{pmatrix} 5.0283 & -0.7730 \\ -0.7730 & 4.7909 \end{pmatrix}$. Hence

$$\Phi^{\frac{5}{3}}\left(A\nabla_\nu B + 2rMm\left(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}\right)\right) - \Phi^{\frac{5}{3}}(A\nabla_\nu B) = \begin{pmatrix} 0.7838 & -1.0172 \\ -1.0172 & 0.7199 \end{pmatrix} > 0.$$

Corollary 2.11. *Let $0 < m \le A, B \le M$ and Φ be a positive unital linear map. Then*

$$\Phi^p\left(\frac{A+B}{2} + Mm(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right) \le \alpha^p\Phi^p\left(A\sharp B\right)$$

and

$$\Phi^p\left(\frac{A+B}{2}+Mm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)\le \alpha^p\left(\Phi\left(A\right)\sharp\Phi\left(B\right)\right)^p.$$

Proof. Take $r=\nu=\frac{1}{2}$ in Theorem 2.7. □

If the positive unital linear map $\Phi(A)=A$ ($A\in\mathbb{B}(\mathscr{H})$), then we get from Theorem 2.7 the following reverse AM–GM inequalities, which improve the reversed AM–GM inequality (1.1).

Corollary 2.12. *Let $0<m\le A,B\le M$. Then, the inequalities*

$$\left(\frac{A+B}{2}+Mm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)^p\le\left(\frac{(M+m)^2}{4Mm}\right)^p(A\sharp B)^p\quad(0<p\le 2)$$

and

$$\left(\frac{A+B}{2}+Mm(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)^p\le\left(\frac{(M+m)^2}{4^{2/p}Mm}\right)^p(A\sharp B)^p\quad(p>2).$$

hold.

The operator Pólya–Szegö inequality states that

$$\Phi(A)\sharp\Phi(B)\le\frac{M+m}{2\sqrt{mM}}\Phi(A\sharp B). \tag{2.8}$$

where $0<m_1^2\le A\le M_1^2$, $0<m_2^2\le B\le M_2^2$, $m=\frac{m_2}{M_1}$ and $M=\frac{M_1}{m_2}$. Also the operator Kantorovich inequality says that

$$\Phi(A)\sharp\Phi(A^{-1})\le\frac{M^2+m^2}{2mM}, \tag{2.9}$$

where $0<m_1^2\le A\le M_1^2$, $0<m_2^2\le B\le M_2^2$, $m=\frac{m_2}{M_1}$, $M=\frac{M_1}{m_2}$; see [6]. In the following result we show some refinements of (2.8) and (2.9).

Theorem 2.13. *Let Φ be a unital positive linear map, $0<m_1^2\le A\le M_1^2$, $0<m_2^2\le B\le M_2^2$, $m=\frac{m_2}{M_1}$, $M=\frac{M_1}{m_2}$.*

$$\Phi(A)\sharp\Phi(B)+\frac{1}{2}\left(\sqrt{Mm}\Phi(A)+\frac{1}{\sqrt{Mm}}\Phi(B)-2\left(\Phi(A)\sharp\Phi(B)\right)\right)\le\frac{M+m}{2\sqrt{mM}}\Phi(A\sharp B). \tag{2.10}$$

In particular, if $B=A^{-1}$, then

$$\Phi(A)\sharp\Phi(A^{-1})+\frac{1}{2}\left(Mm\Phi(A)+\frac{1}{Mm}\Phi(A^{-1})-2\left(\Phi(A)\sharp\Phi(A^{-1})\right)\right)\le\frac{M^2+m^2}{2mM}.$$

Proof. If $0<m_1^2\le A\le M_1^2$ and $0<m_2^2\le B\le M_2^2$, then

$$m^2=\frac{m_2^2}{M_1^2}\le A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\le\frac{M_1^2}{m_2^2}=M^2,$$

whence

$$\left(M - \left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\frac{1}{2}}\right)\left(\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\frac{1}{2}} - m\right) \geq 0.$$

Hence

$$MmA + B \leq (M+m)A\sharp B,$$

whence

$$Mm\Phi(A) + \Phi(B) \leq (M+m)\Phi(A\sharp B). \tag{2.11}$$

Using lemma 2.1 for the operators $Mm\Phi(A)$, $\Phi(B)$ and $\nu = \frac{1}{2}$ we get

$$\sqrt{Mm}\left(\Phi(A)\sharp\Phi(B)\right) + \frac{1}{2}\left(Mm\Phi(A) + \Phi(B) - 2\sqrt{Mm}\left(\Phi(A)\sharp\Phi(B)\right)\right) \leq \frac{1}{2}\left(Mm\Phi(A) + \Phi(B)\right). \tag{2.12}$$

Applying inequalities (2.11) and (2.12) we get the first inequality. In particular, if we consider $m_1^2 = m^2 \leq A \leq M^2 = M_1^2$, then by putting $m_2^2 = \frac{1}{M^2} \leq A^{-1} \leq \frac{1}{m^2} = M_2^2$ in (2.10) we reach the desired inequality. □

If we take Φ in (2.10) to be the positive linear map defined on the diagonal blocks of operators by $\Phi(\mathrm{diag}(A_1, \cdots, A_n)) = \frac{1}{n}\sum_{j=1}^{n} A_j$, then we get the following refinements of a reversed Cauchy-Schwarz operator inequality.

Corollary 2.14. *Let $0 < m_1^2 \leq A_j \leq M_1^2$, $0 < m_2^2 \leq B_j \leq M_2^2$ $(1 \leq j \leq n)$, $m = \frac{m_2}{M_1}$, $M = \frac{M_1}{m_2}$. Then*

$$\left(\sum_{j=1}^{n} A_j\sharp\sum_{j=1}^{n} B_j\right) + \frac{1}{2}\left(\sqrt{Mm}\sum_{j=1}^{n} A_j\frac{1}{\sqrt{Mm}}\sum_{j=1}^{n} B_j - 2\left(\sum_{j=1}^{n} A_j\sharp\sum_{j=1}^{n} B_j\right)\right) \leq \frac{M+m}{2\sqrt{mM}}\left(\sum_{j=1}^{n} A_j\sharp B_j\right).$$

Proposition 2.15. *Let $0 < m \leq A \leq M$ and $x \in \mathscr{H}$. Then*

$$\langle Ax, x\rangle^{\frac{1}{2}}\langle A^{-1}x, x\rangle^{\frac{1}{2}} + \frac{1}{2}\left(\sqrt[4]{Mm}\langle Ax, x\rangle^{\frac{1}{2}} - \frac{1}{\sqrt[4]{Mm}}\langle A^{-1}x, x\rangle^{\frac{1}{2}}\right)^2 \leq \frac{M+m}{2\sqrt{Mm}}\langle x, x\rangle^2.$$

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Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, P.O. Box 98135-674, Zahedan, Iran.

E-mail address: `mojtaba.bakherad@yahoo.com; bakherad@member.ams.org`